Ukrainian Mathematical Journal, Vol. 58, No. 6, 2006

ON THE DYNAMICAL EQUATIONS OF A SYSTEM OF LINEARLY COUPLED NONLINEAR OSCILLATORS

S. N. Bak[1] **and A. A. Pankov**[2]

UDC 517.97

We consider a system of differential equations that describes the dynamics of an infinite chain of linearly coupled nonlinear oscillators. Some results concerning the existence and uniqueness of global solutions of the Cauchy problem are obtained.

1. In the present work, we study equations that describe the dynamics of an infinite chain of linearly coupled nonlinear oscillators. Let q_n be the generalized coordinate of the nth oscillator. If it does not interact with neighboring oscillators, then the equation of its motion is as follows:

$$\ddot{q}_n = -U_n'(q_n), \qquad n \in \mathbb{Z}.$$

We assume that each oscillator interacts linearly with two nearest neighbors. Then the equations of motion of the system considered have the form

$$\ddot{q}_n = -U_n'(q_n) + a_{n-1}(q_{n-1} - q_n) - a_n(q_n - q_{n+1}), \qquad n \in \mathbb{Z}. \tag{1}$$

Equations (1) form an infinite system of ordinary differential equations. We consider solutions of system (1) such that

$$\lim_{n \to \pm\infty} q_n(t) = 0, \tag{2}$$

i.e., the oscillators are at rest at infinity.

Systems of this type are of interest in connection with numerous physical applications [1, 2]. Running waves in such chains were studied in [3], and solutions periodic in time were considered in [4, 5].

In the present work, we consider the problem of the well-posedness of the Cauchy problem for system (1).

2. We write the potential U_n in the form

$$U_n(r) = -\frac{c_n}{2} r^2 + V_n(r)$$

and set

[1] Vinnitsa Pedagogic University, Vinnitsa.

[2] Vinnitsa Pedagogic University, Vinnitsa; College of William and Mary, Williamsburg, USA.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 58, No. 6, pp. 723 – 729, June, 2006. Original article submitted July 13, 2004; revision submitted March 9, 2005.

0041–5995/06/5806–0815

$$b_n = c_n - a_n - a_{n-1}.$$

Then Eq. (1) takes the form

$$\ddot{q}_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n - V_n'(q_n), \qquad n \in \mathbb{Z}. \tag{3}$$

With regard for the boundary conditions (2) and under proper assumptions, this equation can naturally be considered as an operator-differential equation, namely

$$\ddot{q}_n = Aq - B(q), \tag{4}$$

in the Hilbert space l^2 of real two-sided sequences $q = \{q_n\}_{n=-\infty}^{\infty}$; here,

$$(Aq)_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n$$

and the nonlinear operator B is defined as follows:

$$B(q)_n = V_n'(q_n).$$

The scalar product and norm in l^2 are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.

By definition, a solution of Eq. (4) is a twice differentiable function of t with values in l^2.

Assume that the following conditions are satisfied:

(i) the sequences $\{a_n\}$ and $\{c_n\}$ of real numbers are bounded;

(ii) $V_n(r)$ is a function of the class C^1 on $\mathbb{R}$, $V_n(0) = V_n'(0) = 0$, and, for any $R > 0$, there exists $C = C(R) > 0$ such that the following relation holds for all $n \in \mathbb{Z}$:

$$|V_n'(r_1) - V_n'(r_2)| \le C|r_1 - r_2|, \qquad |r_1|,\ |r_2| \le R. \tag{5}$$

It is easy to see that, under these conditions, A is a bounded self-adjoint operator in l^2 and the operator B is bounded and Lipschitz-continuous on every ball of the space l^2. Then, as a consequence of the standard result on local solvability, we conclude that the following theorem is true:

Theorem 1. *Suppose that conditions (i) and (ii) are satisfied. Then, for any* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$, *Eq. (3) has a unique solution of the class* C^2 *defined on a certain interval* $(-t_0; t_0)$ *and satisfying the initial conditions*

$$q(0) = q^{(0)}, \qquad \dot{q}(0) = q^{(1)}. \tag{6}$$

The theorem on global solvability presented below is a consequence of Theorem 1.2 in [6, Chap. 8].

Theorem 2. *Suppose that conditions (i) and (ii), where the constant* C *is independent of* R, *are satisfied. Then, for any* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$, *problem (4), (6) has a unique solution defined for all* $t \in \mathbb{R}$.

3. The conditions of Theorem 3 mean, in particular, that the order of growth of the potential V_n at infinity does not exceed 2. In order to weaken this condition, we note that Eq. (4) can be written in the Hamilton form with the Hamiltonian

$$H(p, q) = \frac{1}{2}\left[\|p\|^2 - (Aq, q) + \sum_{n=-\infty}^{\infty} V_n(q_n)\right],$$

where $p = \dot{q}$. Under conditions (i) and (ii), $H(p, q)$ is a functional of the class C^1 on $l^2 \times l^2$, and a direct calculation shows that H is an integral of Eq. (4), i.e., $H(p(t), q(t))$ does not depend on t for any solution $q(t)$ of Eq. (4).

Theorem 3. *In addition to conditions (i) and (ii), suppose that the operator A is nonpositive, i.e., $(Aq, q) \le 0$ for any $q \in l^2$. Also assume that one of the following two conditions is satisfied:*

(a) $V_n(r) \ge 0$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{R}$;

(b) there exists a nondecreasing function $h(r)$, $r \ge 0$, such that $\lim_{n \to +\infty} h(r) = +\infty$ and $V_n(r) \ge h(|r|)$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{R}$.

Then, for any $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$, problem (4), (6) has a unique solution defined for all $t \in \mathbb{R}$.

Proof. Case (a). Let $q(t)$ be a local solution of problem (4), (6), which exists by virtue of Theorem 1. In order to prove that $q(t)$ is defined on the entire axis, it is sufficient to show that $\|q(t)\| + \|\dot{q}(t)\|$ remains bounded on any finite interval $(-a, a)$ where the solution exists (see, e.g., Theorem X.74 in [7]).

We have

$$H(\dot{q}(t), q(t)) = H(q^{(1)}, q^{(0)}).$$

By virtue of the conditions of Theorem 3 and the definition of Hamiltonian, we get

$$\frac{1}{2}\|\dot{q}(t)\|_{l^2}^2 \le H(q^{(1)}, q^{(0)}).$$

Hence, $\|\dot{q}(t)\|$ is bounded on $(-a, a)$. Since

$$q(t) = \int_0^t \dot{q}(\tau)\,d\tau + q^{(0)},$$

we conclude that $\|q(t)\|$ is bounded.

Case (b). Let $H_0 \ge 0$ be such that $H(q^{(1)}, q^{(0)}) \le H_0$ and let $\bar{r} > 0$ be a solution of the equation $h(r) = H_0$ (it obviously exists). It follows from the definition of H and the conditions of Theorem 3 that $h(|q_n^{(0)}|) \le H_0$ and, hence, $|q_n^{(0)}| \le \bar{r}$. Consider the function $\psi(r)$ defined by the equality

$$\psi(r) = \begin{cases} 1, & 0 \le r \le \bar{r}, \\ -r + \bar{r} + 1, & \bar{r} \le r \le \bar{r} + 1, \\ 0, & r \ge \bar{r} + 1. \end{cases}$$

We set

$$\tilde{V}_n(r) = \int_0^r [\psi(\rho) V_n'(\rho) + (1 - \psi(\rho))] d\rho.$$

It is easy to verify that the modified equation (3) with potential $\tilde{V}_n$ satisfies the conditions of Theorem 2 and, hence, has a global solution $q(t)$ with initial data $q^{(0)}$ and $q^{(1)}$. By elementary calculations, we obtain $\tilde{V}_n(r) \ge h(r)$, where

$$\tilde{h}(r) = \begin{cases} h(r), & 0 \le r \le \bar{r}, \\ (\bar{r} + 1 - r) h(r) + \int_{\bar{r}}^{r} h(\rho) d\rho + \left(\frac{r^3}{3} - \bar{r} \frac{r^2}{2} + \frac{\bar{r}^3}{6} \right), & \bar{r} \le r \le \bar{r} + 1, \\ r^2 + \int_{\bar{r}}^{\bar{r}+1} h(\rho) d\rho + \left[\frac{(\bar{r}+1)^2}{3} + \frac{\bar{r}^3}{6} \right], & r \ge \bar{r} + 1. \end{cases}$$

For the modified Hamiltonian $\tilde{H}$, we get $\tilde{H}(p(t), q(t)) = \tilde{H}(q^{(1)}, q^{(0)})$. Since $|q_n^{(0)}| \le \bar{r}$, we conclude that $\tilde{H}(q^{(1)}, q^{(0)}) \le H_0$. Hence, $\tilde{h}(|q_n|) \le H_0$. Further, since $\tilde{h}(r) \ge \tilde{h}(\tilde{r}) = h(\tilde{r}) = H_0$, we have $|q_n| \le \tilde{r}$. Taking into account that the modified equation coincides with the original one for $q(t)$, we complete the proof of Theorem 3.

Under certain additional assumptions, the nonpositivity conditions in Theorem 3 can be omitted.

Corollary 1. *Suppose that the conditions of Theorem 3 [case (b)] except the condition of the nonpositivity of the operator A are satisfied and, furthermore,* $\lim_{r \to +\infty} h(r)/r^2 = +\infty$. *Then problem (4), (6) has a unique global solution for any $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$.*

Proof. We represent U_n in the form

$$U_n(r) = -\frac{c_n - 2\lambda}{2} r^2 + (V_n(r) - \lambda r^2),$$

where $\lambda > 0$ is sufficiently large. Then the new operator A corresponding to the coefficients a_n and $c_n - 2\lambda$ is nonpositive. At the same time, we have

$$V_n(r) - \lambda r^2 \ge h(|r|) - \lambda r^2 = h(|r|) \left(1 - \lambda \frac{r^2}{h(|r|)} \right).$$

Hence,

$$V_n(r) - \lambda r^2 \geq k_1 h(|r|) - k_2,$$

where $k_1 \in (0, 1)$ and $k_2 \geq 0$. It remains to use Theorem 3 with $h(r)$ replaced by $k_1 h(r) - k_2$.

Corollary 1 is proved.

Corollary 1 can be applied, e.g., to potentials of the form $V_n(r) = \alpha_n r^3 + \beta_n r^4$, where the sequences α_n and β_n are bounded and $\beta_n \geq \kappa > 0$.

4. The arguments of Sec. 3 are also valid for singular potentials of the Lennard-Jones type [8]. We preserve condition (i) and replace condition (ii) by the following one:

(iii) the function $V_n(r)$ belongs to C^1 on $(-\infty, d)$, $d > 0$, and, on every finite interval $[\alpha, \beta] \subset (-\infty, d)$, inequality (5), where the constant C does not depend on $n \in \mathbb{Z}$ but possibly depends on the interval, is true.

Theorem 4. *Suppose that conditions (i) and (iii) are satisfied, the operator A is nonpositive, and there exists a function $h(r)$ on $(-\infty, r)$ that does not increase on some interval $(-\infty, \alpha_0)$, does not decrease on $(\alpha_0\ d)$, and satisfies the conditions $\lim_{r \to d} h(r) = \lim_{r \to -\infty} h(r) = +\infty$ and $V_n(r) \geq h(r)$ for all $n \in \mathbb{Z}$ and $r \in (-\infty, d)$. Then problem (4), (6) has a unique global solution for all $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$ such that $q_n^{(0)} < d$.*

The proof of Theorem 4 is analogous to that of Theorem 3 [case (b)]. The potential $V_n(r)$ is glued with the quadratic potential on some intervals $(\alpha_0, \alpha_0 + \varepsilon)$ and $(-\beta_0, -\beta_0 - 1)$. Applying Theorem 2 to the modified equation, we verify that its solution is indeed a solution of the original equation.

Note that the solutions $q_n(t)$ satisfy the inequality $q_n(t) < d$ for all $t \in \mathbb{R}$. Since the set $\{q \in l^2 : q_n < d,\ n \in \mathbb{Z}\}$ is not open in l^2, the classical results on the local solvability cannot be used in the case under consideration.

5. Now consider the case

$$V_n(r) = \frac{d_n}{3} r^3,$$

where d_n is a bounded sequence. Assume that the operator A is negative definite, i.e., $(Aq, q) \leq -\alpha_0 \|q\|^2$, $\alpha_0 > 0$, for $q \in l^2$.

We set

$$J(q) = -\frac{1}{2}(Aq, q) + \frac{1}{3}\sum_{n \in \mathbb{Z}} d_n q_n^3 = \frac{1}{2}a(q) + \frac{1}{3}b(q).$$

Note that $a(q)^{1/2}$ is the norm in l^2 equivalent to the standard one. Then

$$H(p,q) = \frac{1}{2}\|p\|^2 + J(q).$$

Since $|b(q)| \le c'\|q\|_{l^3}^3 \le c''\|q\|^3$, there exists a constant $c > 0$ such that

$$|b(q)|^{1/3} \le ca(q)^{1/2}, \qquad q \in l^2. \tag{7}$$

We set

$$\gamma = \inf\left\{\sup_{\lambda \ge 0} J(\lambda q) : q \in l^2, \ q \ne 0\right\}. \tag{8}$$

Lemma 1. $\gamma \ge 1/(6c^6)$.

Proof. We have

$$J(\lambda q) = \frac{\lambda^2}{2}a(q) + \frac{\lambda^3}{3}b(q).$$

If $b(q) \ge 0$, then

$$\sup_{\lambda \ge 0} J(\lambda q) = +\infty,$$

and if $b(q) < 0$, then

$$\sup_{\lambda \ge 0} J(\lambda q) = J\left(-\frac{a(q)}{b(q)}q\right) = \frac{1}{6}\frac{a(q)^3}{b(q)^2}.$$

Inequality (7) yields the required result.

We set

$$W_\gamma = \left\{q \in l^2 : 0 \le J(\lambda q) < \gamma \ \ \forall \lambda \in [0,1]\right\}. \tag{9}$$

It is obvious that W_γ is star-shaped with respect to the origin, i.e., if $q \in W_\gamma$, then $\theta q \in W_\gamma$ for any $\theta \in [0,1]$.

Lemma 2. *The set* W_γ *contains an open ellipsoid*

$$B = \left\{q \in l^2 : a(q) < \rho\right\}$$

for any ρ *such that* $\rho \le 9/(4c^2)$ *and* $\rho/2 + (c^3/3)\rho^{3/2} < \gamma$.

Proof. By virtue of (7), we have

$$\frac{\lambda^2}{2}a(q) - \frac{\lambda^3 c^3}{3}a(q)^{3/2} \le J(\lambda q) \le \frac{\lambda^2}{2}a(q) + \frac{\lambda^3 c^3}{3}a(q)^{3/2}.$$

Hence, $J(\lambda q) \ge 0$ for any $\lambda \in [0,1]$, provided that

$$\frac{1}{2} - \frac{\lambda c^3}{3}a(q)^{1/2} \ge 0 \qquad \forall \lambda \in [0,1].$$

Therefore, if $a(q) \le 9/(4c^2)$, then $J(\lambda q) < \gamma$ by virtue of the second condition for ρ.

Lemma 2 is proved.

We set

$$W_{*,\gamma} = \left\{ q \in l^2 : a(q) + b(q) > 0, \ J(q) < \gamma \right\}.$$

It follows from the continuity of functionals $a(q)$ and $b(q)$ that $W_{*,\gamma}$ is an open set.

Lemma 3. $W_\gamma = W_{*,\gamma} \cup B$.

Proof. It suffices to show that $W_\gamma = W_{*,\gamma} \cup \{0\}$.

Let $q \in W_\gamma$, $q \ne 0$. If $b(q) \ge 0$, then $a(q) + b(q) > 0$ and $J(q) < \gamma$. If $b(q) < 0$, then

$$\sup J(\lambda q) = J\left(-\frac{a(q)}{b(q)}q\right) \ge \gamma.$$

Therefore, $-a(q)/b(q) > 1$ and $J(q) < \gamma$, whence $q \in W_{*,\gamma}$.

Conversely, let $q \in W_{*,\gamma}$. If $b(q) \ge 0$, then

$$\sup_{\lambda \in [0,1]} J(\lambda q) = J(q) < \gamma$$

and $q \in W_\gamma$. If $b(q) < 0$, then the inequality $-a(q)/b(q) > 1$ yields

$$\sup_{\lambda \in [0,1]} J(\lambda q) = J(q),$$

which gives the required result.

By virtue of the openness of $W_{*,\gamma}$ and B, Lemma 3 implies that the set W_γ is open, i.e., it is a neighborhood of zero in l^2.

Lemma 4. *W_γ is a bounded set.*

Proof. If $b(q) \geq 0$, then $J(q) \geq a(q)/2$ and $a(q) < 2\gamma$. If $b(q) < 0$, then, according to Lemma 3, we have $b(q) > -a(q)$. Hence, $J(q) > a(q)/6$ and $a(q) < 6\gamma$. Thus, W_γ is contained in the bounded set $\{q \in l^2 : a(q) < 6\gamma\}$.

Theorem 5. *Suppose that* $V_n(r) = (d_n/3)r^3$, *where* d_n *is a bounded sequence, the operator* A *is negative definite, and* $q^{(0)} \in W_\gamma$ *and* $q^{(1)} \in l^2$ *are such that*

$$\frac{1}{2}\|q^{(1)}\|^2 + J(q^{(0)}) < \gamma.$$

Then the Cauchy problem with initial data $q^{(0)}$ *and* $q^{(1)}$ *has a unique global solution.*

Proof. The existence and uniqueness of a local solution $q(t)$ follow from Theorem 1. As in the proof of Theorem 3 [case (a)], it suffices to show that $q(t)$ remains bounded.

Let us show that $q(t) \in W_\gamma$. Assume that this is not true and $t_1 > 0$ is the least value of $t > 0$ for which $q(t_1) \notin W_\gamma$. Then $q(t_1)$ belongs to the boundary ∂W_γ of the set W_γ. Since W_γ is star-shaped, we have $\theta q(t_1) \in W_\gamma$ for any $\theta \in [0,1)$. Hence, $J(\theta q(t_1)) < \gamma$. Passing to the limit as $\theta \to 1$, we get $J(q(t_1)) \leq \gamma$. If $J(q(t_1)) < \gamma$, then, by virtue of the definition of W_γ and the inequality $J(\theta q(t_1)) < \gamma$, we have $q(t_1) \in W_\gamma$, which contradicts the assumption made above. Thus, $J(q(t_1)) = \gamma$.

Since the Hamiltonian H is invariant, we have

$$J(q(t_1)) \leq \frac{1}{2}|\dot{q}(t_1)|^2 + J(q(t_1)) = \frac{1}{2}|q^{(1)}|^2 + J(q^{(0)}) < \gamma.$$

The contradiction obtained shows that $q(t) \in W_\gamma$ for all $t > 0$ for which q is defined. Therefore, the solution exists for all $t > 0$.

Since Eq. (1) is invariant under the replacement of t by $-t$, the solution is defined for all $t \in \mathbb{R}$.

Theorem 5 is a discrete analog of a result of [9] related to a nonlinear wave equation. In particular, this theorem yields the global solvability of the Cauchy problem in the case where the initial data are sufficiently small in the l^2-norm.

REFERENCES

1. S. Aubry, “Breathers in nonlinear lattices: existence, linear stability and quantization,” *Physica D,* **103**, 201–250 (1997).
2. O. M. Braun and Y. S. Kivshar, “Nonlinear dynamics of the Frenkel–Kontorova model,” *Phys. Rept.,* **306**, 1–108 (1998).
3. G. Iooss and K. Kirchgassner, “Traveling waves in a chain of coupled nonlinear oscillators,” *Commun. Math. Phys.,* **211**, 439–464 (2000).
4. S. N. Bak and A. A. Pankov, “On periodic oscillations of an infinite chain of linearly coupled nonlinear oscillators,” *Dopov. Nats. Akad. Nauk Ukr.,* No. 9, 13–16 (2004).
5. S. N. Bak, “A method of conditional minimization in the problem of oscillations of a chain of nonlinear oscillators,” *Mat. Fiz., Analiz, Geometr.,* No. 3, 263–273 (2004).
6. Yu. L. Daletskii and M. G. Krein, *Stability of Solutions of Differential Equations in a Banach Space* [in Russian], Nauka, Moscow (1970).
7. M. Reed and B. Simon, *Methods of Modern Mathematical Physics,* Vol. 2, Academic Press, New York (1975).
8. G. Friesecke and J. Wattis, “Existence theorem for solitary waves on lattices,” *Commun. Math. Phys.,* **161**, 391–418 (1994).
9. D. Sattinger, “On global solutions of nonlinear hyperbolic equations, “ *Arch. Rat. Mech. Anal.,* **30**, 148–172 (1968).