Communications in Mathematical Analysis *www.commun-math-anal.org*
Volume 3, Number 1, pp. 19–26 (2007)

PERIODIC TRAVELING WAVES IN CHAINS OF OSCILLATORS

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(Communicated by Toka Diagana)

Abstract

It is considered the system of differential equations that describes the dynamics of an infinite chain of linearly coupled nonlinear oscillators. Results on existence of the periodic travelling waves are obtained.

AMS Subject Classification: 39A12; 39A70; 58E50.

Keywords: chain of oscillators, traveling wave, critical point, linking.

1 Introduction

In the present paper we study equations that describes the dynamics of an infinite chain of linearly coupled nonlinear oscillators. Let q_n be a generalized coordinate of the n-th oscillator. In absence of interaction between the oscillators equations of motion are

$$\ddot{q}_n = -U_n'(q_n), n \in \mathbb{Z}.$$

We assume that each oscillator interacts linearly with two nearest neighbors. Then the equations of motion of the system considered have the form

$$\ddot{q}_n = -U_n'(q_n) + a_{n-1}(q_{n-1} - q_n) - a_n(q_n - q_{n+1}), n \in \mathbb{Z}. \tag{1.1}$$

Equations (1.1) form an infinite system of ordinary differential equations.

Systems of such type are of interest in connection with numerous physical applications [1], [5], [6]. In the paper [8] traveling waves in chains of the form (1.1) are studied by means of bifurcation theory, while [1], [2], [3] and [9] deal with periodic in time solutions. The survey of known results in this direction can be found in [10].

In the present paper we obtain, by means of the linking theorem, a result on the existence of periodic traveling waves. Actually, this result extends our previous existence theorem [4] to a wider range of speed.

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2 Statement of a problem

Let us consider a homogeneous on spatial variable chain of linearly coupled nonlinear oscillators with potential

$$U_n(r) = U(r) = -\frac{c_0}{2}r^2 + V(r).$$

Then the corresponding equations of motion read

$$\ddot{q}_n = a\triangle_d q_n + c_0 q_n - V'(q_n), \tag{2.1}$$

where

$$(\triangle_d q)_n = q_{n+1} + q_{n-1} - 2q_n,$$

$\triangle_d$ is the 1-dimensional discrete Laplacian.

Traveling wave is a solution of the form

$$q_n(s) = u(n - cs),$$

where the function $u(s), s \in \mathbb{R}$, is called the profile function, or simply profile, of the wave and the constant $c \neq 0$ the speed of the wave. If $c > 0$ then the wave moves to right, otherwise to the left.

Making use the traveling wave Ansatz we obtain the equation

$$c^2u''(s) = a(u(s+1) + u(s-1) - 2u(s)) + c_0u(s) - V'(u(s)) \tag{2.2}$$

for the profile function $u(s)$. This equation has, actually, a variational structure.

A periodic traveling wave is a traveling wave such that its profile function is a periodic function, i.e.

$$u(s+2k) = u(s), t \in \mathbb{R}, \tag{2.3}$$

where k is an arbitrary positive real number.

In what follows we consider solutions of equation (1.1) that are C^2-functions.

3 Variational setting

We always assume that

(h) *the function* $V(r)$ *is* C^1, $V(0) = V'(0) = 0$, $V'(r) = o(r)$ *as* $r \to 0$ *and there exists* $\mu > 2$ *such that*

$$0 < \mu V(r) \leq V'(r)r, r \neq 0.$$

Let

$$E_k = \{u \in H^1_{loc}(\mathbb{R}) : u(s+2k) = u(s)\}$$

endowed with the norm

$$\|u\|_k = (\|u\|^2_{L^2(-k,k)} + \|u'\|^2_{L^2(-k,k)})^{1/2} = (\int_{-k}^{k}(|u(s)|^2 + |u'(s)|^2)ds)^{1/2},$$

i.e. the Sobolev space of $2k$-periodic functions. On this space we consider the functional

$$J_k(u) = \int_{-k}^{k} \{ \frac{c^2}{2}(u'(t))^2 - \frac{a}{2}(u(t+1)-u(t))^2 + \frac{c_0}{2}u^2(t) - V(u(t))\} dt. \tag{3.1}$$

The following simple lemmas can be found in [4] (see also [10] in the case of Fermi–Pasta–Ulam lattices).

For simplicity denote

$$(Au)(s) := u(s+1) - u(s).$$

Lemma 3.1. *Under assumption* (h) *the functional* J_k *is* C^1 *on* E_k, *and*

$$\langle J_k'(u), h\rangle = \int_{-k}^{k} \{c^2 u'(s)h'(s) + a(u(s+1)+$$

$$+u(s-1) - 2u(s))h(s) + c_0 u(s)h(s) - V'(u(s))h(s)\} ds \tag{3.2}$$

for $u, h \in E_k$.

Lemma 3.2. *Any critical point of* J_k *is* C^2*-solution of equation (2.2)satisfying (2.3).*

Lemma 3.3. *We have*

$$\|Au\|_{L^2} \le \|u'\|_{L^2}$$

and

$$\|Au\|_{L^2} \le 2\|u\|_{L^2} \tag{3.3}$$

for all $u \in E_k$.

4 Main results

Making use of the linking theorem, we shall prove the existence of nontrivial traveling waves with periodic profile function. For, due to a Lemma 3.2, it is enough to prove the existence of a nontrivial critical point of J_k.

Theorem 4.1. *Assume* (h). *Suppose that* $c_0 > 0$. *Then for every* $k \ge 1$ *and* $c > 0$ *equation (2.2) has a nontrivial solution* u, *that satisfies (2.3), i.e. there exist two* $2k$*-periodic traveling waves with the speed* $\pm c$.

Let us formulate the linking theorem ([10], [11], [13]).

Let H be a Hilbert space, $H = Y \oplus Z$, where $dimY < \infty$. Let $\rho > r > 0$ and $z \in Z : \|z\| = r$. Define

$$M := \{u = y + \lambda z : y \in Y, \|u\| \le \rho, \lambda \ge 0\}$$

and

$$M_0 := \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \ge 0, \text{ or } \|u\| \le \rho \text{ and } \lambda = 0\},$$

i.e. $M_0 = \partial M$ is the boundary M. Let

$$N := \{u \in Z : \|u\| = r\}.$$

Consider a functional φ on H and suppose that

$$\beta := \inf_{u\in N} \varphi(u) > \alpha := \sup_{u\in M_0} \varphi(u).$$

In this situation we say that the functional φ possesses the linking geometry.

Theorem 4.2 (Linking)**.** *Suppose that the functional φ of class C^1 possesses the linking geometry and satisfies the Palais–Smale condition*

(PS) *if any sequence $u_n \in H$ such that $\varphi'(u_n) \to 0$ and the sequence $\varphi(u_n)$ is bounded then (u_n) contains a convergent subsequence.*

Let

$$b := \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi(\gamma(u)),$$

where

$$\Gamma := \{\gamma \in C(M;H) : \gamma = id \text{ on } M_0\}.$$

Then b is critical value of φ and

$$\beta \le b \le \sup_{u\in M} \varphi(u).$$

Let us begin with verifying condition (PS).

Lemma 4.3. *Under assumptions of Theorem 4.1 functional J_k satisfies the Palais–Smale condition.*

Proof. Let $u_n \in E_k$ be a Palais–Smale sequence at the level b, i.e. $J_k(u_n) \to b$. Choose $\beta \in (\mu^{-1}, 2^{-1})$. Then for n large we have

$$b + 1 + \beta\|u_n\|_k \ge J_k(u_n) - \beta\langle J_k'(u_n), u_n\rangle =$$

$$= (\frac{1}{2} - \beta)\int_{-k}^{k} (c^2|u_n'|^2 - a|Au_n|^2 + c_0|u_n|^2)ds -$$

$$- \int_{-k}^{k} (V(u_n) - \beta V'(u_n)u_n))ds.$$

If $a \le 0$, then

$$J_k(u_n) - \beta\langle J_k'(u_n), u_n\rangle \ge$$

$$\ge (\frac{1}{2} - \beta)\int_{-k}^{k} (c^2|u_n'|^2 + c_0|u_n|^2)ds \ge (\frac{1}{2} - \beta)\alpha_0\|u_n\|_k^2,$$

where $\alpha_0 = \min\{c^2; c_0\}$. Hence,

$$b + 1 + \beta\|u_n\|_k \ge (\frac{1}{2} - \beta)\alpha_0\|u_n\|_k^2,$$

and this implies immediately that u_n is bounded in E_k.

If $a > 0$, then

$$J_k(u_n) - \beta\langle J_k'(u_n), u_n\rangle \geq$$

$$\geq (\frac{1}{2} - \beta)(c^2\|u_n'\|^2_{L^2} - a\|Au_n\|^2_{L^2} + c_0\|u_n\|^2_{L^2}) +$$

$$+ C(\beta\mu - 1)\|u_n\|^{\mu}_{L^{\mu}} - C_0.$$

Since $\mu > 2$, we have, by Lemma 3.3,

$$\|Au_n\|^2_{L^2} \leq 4\|u_n\|^2_{L^2} \leq C\|u_n\|^2_{L^{\mu}} \leq K(\varepsilon) + \varepsilon\|u_n\|^{\mu}_{L^{\mu}},$$

where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Then

$$b + 1 + \beta\|u_n\|_k \geq (\frac{1}{2} - \beta)c^2\|u_n'\|^2_{L^2} + (\frac{1}{2} - \beta)c_0\|u_n\|^2_{L^2} -$$

$$-(\frac{1}{2} - \beta)a\varepsilon\|u_n\|^{\mu}_{L^{\mu}} - (\frac{1}{2} - \beta)aK(\varepsilon) +$$

$$+ C(\beta\mu - 1)\|u_n\|^{\mu}_{L^{\mu}} - C_0.$$

Choosing ε small enough, we have

$$b + 1 + \beta\|u_n\|_k \geq (\frac{1}{2} - \beta)(c^2\|u_n'\|^2_{L^2} + c_0\|u_n\|^2_{L^2}) +$$

$$+ C_1\|u_n\|^{\mu}_{L^{\mu}} - C_0 \geq$$

$$\geq (\frac{1}{2} - \beta)\alpha_1\|u_n\|^2_k + C_1\|u_n\|^{\mu}_{L^{\mu}} - C_0,$$

where $\alpha_1 = \min\{c^2; c_0\}$. Since $\beta\mu - 1 > 0$ then $C_1 > 0$, and we have

$$b + 1 + \beta\|u_n\|_k \geq (\frac{1}{2} - \beta)\alpha_1\|u_n\|^2_k - C_0.$$

The last inequality implies that u_n is bounded.

Since u_n is bounded in Hilbert space E_k then, up to a subsequence, we have that $u_n \to u$ weakly in E_k, hence, $Au_n \to Au$ weakly in E_k. By the compactness of Sobolev embedding, both these convergences are strong in $L^2(-k;k)$ and in $C([-k,k])$. A straightforward calculation shows that

$$c^2\|u_n - u\|^2_k = \int_{-k}^{k}(c^2|u_n' - u'|^2 + c^2|u_n - u|^2)ds =$$

$$= \langle J_k'(u_n) - J_k'(u), u_n - u\rangle + a\|Au_n - Au\|^2_{L^2} -$$

$$-c_0\|u_n - u\|^2_{L^2} + \int_{-k}^{k}(V'(u_n) - V'(u))(u_n - u)ds.$$

Obviously that all the terms on the right converge to 0 (first and last terms converge to 0 by weak convergence, second and third terms converge to 0 by strong convergence). Therefore, we conclude that $\|u_n - u\| \to 0$ that proves the lemma. □

Lemma 4.4. *If V satisfies (h) then there exist constants $d > 0$ and $d_0 \geq 0$, that*

$$V(r) \geq d|r|^{\mu} - d_0. \tag{4.1}$$

Proof. Let fix $r_0 > 0$. Since

$$V'(r) \geq \mu \frac{V(r)}{r}$$

then, by standard results for differential inequalities [7], $V(r) \geq y(r)$ as $r \geq r_0$, where $y(r)$ is solution of differential equation

$$y'(r) = \frac{\mu}{r} y(r)$$

with initial data $y(r_0) = V(r_0)$. Obviously,

$$y(r) = \frac{V(r_0)}{r_0^{\mu}} r^{\mu}.$$

Hence,

$$V(r) \geq \frac{V(r_0)}{r_0^{\mu}} r^{\mu}, r \geq r_0.$$

Then for all $r \geq 0$

$$V(r) \geq V(r_0)(\frac{r^{\mu}}{r_0^{\mu}} - 1) = \frac{V(r_0)}{r_0^{\mu}} r^{\mu} - V(r_0).$$

Similarly, for $r \leq 0$

$$V(r) \geq \frac{V(-r_0)}{r_0^{\mu}} |r^{\mu}| - V(r_0).$$

Thus, we obtain (4.1) with

$$d = \min[\frac{V(-r_0)}{r_0^{\mu}}, \frac{V(r_0)}{r_0^{\mu}}],$$

$$d_0 = \max[V(r_0), V(-r_0)].\square$$

Lemma 4.5. *Under assumptions of Theorem 4.1 functional J_k possesses the linking geometry.*

Proof. Consider the operator L defined by

$$(Lu)(s) := -c^2 u''(s) + a(u(s+1) + u(s-1) - 2u(s)) + c_0 u(s).$$

Elementary Fourier analysis shows that L is a self-adjoint operator in $L^2(-k;k)$, bounded below and that L has discrete spectrum which accumulated at $+\infty$. The eigenvalues and eigenfunctions can be calculated explicity, but we do not use this fact. We mention only that all eigenvalues, λ_j, with nonconstant eigenfunctions are double. Denote by $h_j^{\pm} \in E_k$ linearly independent pairs of eigenfunctions with the eigenvalues λ_j.

Let Z be the subspace of E_k generated by the functions $h_j^{\pm}$ with $\lambda_j > 0$ and Y be subspace of E_k generated by the functions $h_j^{\pm}$ with $\lambda_j \leq 0$. Note that $dim Y < \infty$. It is readily verified that $Y \perp Z$ and $E_k = Y \oplus Z$.

Denote by Q_k the quadratic part of the functional J_k

$$Q_k(u) = \frac{1}{2}\int_{-k}^{k}(c^2|u'|^2 - a|Au|^2 + c_0|u|^2)ds.$$

Obviously,

$$Q_k(y+z) = Q_k(y) + Q_k(z),$$

where $y \in Y, z \in Z$.

Note, that on Z the quadratic form Q_k is positive, i.e.

$$Q_k(u) \geq \alpha \|u\|_k^2,$$

where $\alpha > 0$. Assumption (h) implies that, given $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$|V(r)| \leq \varepsilon r^2,$$

if $|r| \leq r_0$. Then

$$J_k(u) \geq Q_k(u) - \varepsilon \int_{-k}^{k} |u|^2 ds \geq Q_k(u) - \varepsilon \|u\|_k^2 \geq \delta \|u\|_k^2,$$

where $\delta > 0$. Hence,

$$J_k(u) > 0$$

on $N = \{u \in Z : \|u\|_k = r\}$ provided $r > 0$ is small enough.

Now we fix any $z \in Z, \|z\|_k = 1$, and set

$$M = \{u = y + \lambda z : y \in Y, \|u\|_k \leq \rho, \lambda \leq 0\}.$$

We have to prove that $J_k(u) \leq 0$ on $M_0 = \partial M$ provided ρ is large enough.

Recall that

$$M_0 = \{u = y + \lambda z : y \in Y, \|u\|_k = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\|_k \leq \rho \text{ and } \lambda = 0\}.$$

We have

$$J_k(y+\lambda z) = Q_k(y) + \lambda^2 Q_k(z) - \int_{-k}^{k} V(y+\lambda z)ds.$$

By Lemma 4.4, there exist constants $d > 0$ and $d_0 > 0$ such that

$$V(r) \geq d|r|^{\mu} - d_0,$$

where $\mu > 2$. Then, since $Q_k(y) \leq 0$, we have

$$J_k(y+\lambda z) \leq \lambda^2 \gamma_0 + 2kd_0 - d\|y+\lambda z\|_{L^{\mu}}^{\mu},$$

where $\gamma_0 = Q_k(z)$. Since

$$\rho^2 = \|y+\lambda z\|_k^2 = \|y\|_k^2 + \lambda^2,$$

we have $\lambda^2 \leq \rho^2$. Furthermore, on finite dimensional spaces all norms are equivalent. Hence,

$$\|y+\lambda z\|_{L^{\mu}} \geq c\|y+\lambda z\|_k = c\rho$$

and

$$J_k(y+\lambda z) \le \gamma_0\rho^2 + 2kd_0 - d\rho^\mu.$$

Since $\mu > 2$, the right hand part here is negative if ρ is large enough. Hence, $J_k(y+\lambda z) \le 0$. If $u \in M_0, \|u\|_k \le \rho$ and $\lambda = 0$, then $u = y \in Y$ and, obviously, $J_k(u) \le 0$. Thus, we see that J_k possesses linking geometry. □

Proof of Theorem 4.1. Due to Lemma 4.3 and Lemma 4.5, functional J_k satisfies all conditions of linking theorem. Hence, J_k has a nontrivial critical point $u \in E_k$. By Lemma 3.2, u is a C^2-solution of equation (2.2) that satisfy (2.3). The proof is complete. □

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