PHYSICAL AND MATHEMATICAL SCIENCES

ON WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR SYSTEM OF OSCILLATORS IN WEIGHTED SPACES

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We study equations that describe the dynamics of an infinite system of linearly coupled nonlinear oscillators on a two dimensional lattice. Let $q_{n,m}=q_{n,m}(t)$ be a generalized coordinate of the (*n, m*)-th oscillator at time t. It is assumed that each oscillator interacts linearly with its four nearest neighbors. The equations of motion of the system is of the form

$$\ddot{q}_{n,\,m}=a_{n-1,\,m}q_{n-1,\,m}+a_{n,\,m}q_{n+1,\,m}+b_{n,\,m-1}q_{n,\,m-1}+b_{n,\,m}q_{n,\,m+1}+ \\ +c_{n,\,m}q_{n,\,m}-V'_{n,\,m}(q_{n,\,m}),\ (n,m)\in Z^{2}. \tag{1}$$

We consider solutions of system (1) such that

$$\lim_{n,\,m\to\pm\infty} q_{n,\,m}(t)=0, \tag{2}$$

i.e., the oscillators are at the rest at infinity.

We study the Cauchy problem for system (1) with initial conditions

$$q_{n,m}(0)=q^{(0)}_{n,m},\ \dot{q}_{n,m}(0)=q^{(1)}_{n,m}. \tag{3}$$

where $q^{(0)}_{n,m}$ and $q^{(1)}_{n,m}$ are given real sequences. System (1) naturally can be considered as an operator-differential equation, namelly

$$(Aq)_{n,\,m}=a_{n-1,\,m}q_{n-1,\,m}+a_{n,\,m}q_{n+1,\,m}+b_{n,\,m-1}q_{n,\,m-1}+b_{n,\,m}q_{n,\,m+1}+c_{n,\,m}q_{n,\,m}, \tag{4}$$

where

$$(Aq)_{n,\,m}=a_{n-1,\,m}q_{n-1,\,m}+a_{n,\,m}q_{n+1,\,m}+b_{n,\,m-1}q_{n,\,m-1}+b_{n,\,m}q_{n,\,m+1}+c_{n,\,m}q_{n,\,m},$$

$$(B(q))_{n,\,m}=V'_{n,\,m}(q_{n,\,m}),$$

in the Hilbert, or even Banach, space E of sequences. Within this framework, initial conditions (3) become

$$q(0)=q^{(0)},\ \dot{q}(0)=q^{(1)}. \tag{5}$$

Throughout the paper we impose the following assumptions

(i) $\{a_{n,m}\}$, $\{b_{n,m}\}$, $\{d_{n,m}\}\subset R$ *are bounded;*

(ii) $V_{n,m}(r)\in C^1(R)$, $V_{n,m}(0)=V'_{n,m}(0)=0$ *and for any* $R>0$ *there exists* $C=C(R)>0$ *such that* $\forall\,(n,m)\in Z^2$:

$$\left|V'_{n,m}(r_1)-V'_{n,m}(r_2)\right|\le C|r_1-r_2|,\quad |r_1|,|r_2|\le R.$$

Sometimes we use the following stronger than (ii) assumption:

(ii') *assumption* (ii) *is satisfied with the constant* $C>0$ *independent of* R.

Let $\Theta=\{\theta_{n,m}\}$ be a sequence of positive numbers (weight). We denote by l^2_Θ the space of all two-sided sequences $q=\{q_{n,m}\}$ of real numbers such that the norm

$$\|q\|_\Theta=\left(\sum_{(n,m)\in Z^2}\theta_{n,m}|q_{n,m}|^2\right)^{\frac{1}{2}}$$

is finite. This is a Hilbert space with the scalar product

$$\left(q^{(1)},q^{(2)}\right)_\Theta=\sum_{(n,m)\in Z^2}\theta_{n,m}q^{(1)}_{n,m}q^{(2)}_{n,m}.$$

If $\theta_{n,m}\equiv 1$, then $l^2_\Theta=l^2$.

We suppose that the weight satisfies the following assumption

(iii) *the weight be a regular, i.e., the sequence* $\Theta=\{\theta_{n,m}\}$ *is bounded below by a positive constant and there exists a constant* $c_0>0$ *such that*

$$c_0^{-1}\le\frac{\theta_{n+1,m}}{\theta_{n,m}},\ \frac{\theta_{n,m+1}}{\theta_{n,m}}\le c_0$$

for all $(n,m)\in Z^2$.

We obtain the following results.

Theorem 1. *Assume* (i), (ii') *and* (iii). *Then for every* $q^{(0)}\in l^2_\Theta$ *and* $q^{(1)}\in l^2_\Theta$ *problem (1), (3) has a unique global solution* $q\in C^2(R;l^2_\Theta)$.

Theorem 2. *Assume* (i), (ii') *and* (iii). *Suppose that the operator* A *is non-ositive, i.e.,* $(Aq,q)\le 0$ *for all* $q\in l^2$. *Suppose also that one of the following two conditions holds:*

(a) $V_{n,m}(r)\ge 0$ *for all* $(n,m)\in Z^2$ *and* $r\in R$;

(b) *there exists a nondecreasing function* $h(\xi)$, $\xi\ge 0$, *such that* $\lim_{\xi\to+\infty}h(\xi)=+\infty$ *and* $V_{n,m}(r)\ge h(|r|)$ *for all* $(n,m)\in Z^2$ *and* $r\in R$.

Then for every $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$ *problem (1), (3) has a unique global solution* $q \in C^2\left(R\,;\,l^2_\Theta\right)$.

References

1. Aubry S. Breathers in nonlinear lattices: Existence, linear stability and quantization. *Physica D*. 1997. Vol. 103. P. 201-250.

2. Bak S. M. Existence of heteroclinic traveling waves in a system of oscillators on a twodimensional lattice. *J. Math. Sci.* 2016. Vol. 217, № 2 (August). P. 187-197. doi:10.1007/s10958-016-2966-z.

3. Bak S. M. Existence of the solitary traveling waves for a system of nonlinearly coupled oscillators on the 2d-lattice. *Ukr. Math. J.* 2017. Vol. 69 (2017), № 4 (September). P. 509-520. doi: 10.1007/s11253-017-1378-7**.**

4. Bak S. M. Existence of the time periodic solutions of system of oscillators on 2D-lattice. *Carpathian Mathematical Publications*. 2012. Vol. 4, № 2. P. 5-12. (in Ukrainian)

5. Bak S. M. Global well-posedness of the Cauchy problem for system of oscillators on 2D-lattice with power potentials. *Journal of Mathematical Sciences*. 2020. Vol. 246, № 5 (May). P. 593-601. doi:10.1007/ s10958-020-04765-6**.**

6. Bak S. M. Homoclinic traveling waves in discrete sine–Gordon equation with nonlinear interaction on 2D lattice. *Matematychni Studii*. 2019. Vol. 52, № 2. P. 176-184. doi:10.30970/ms.52.2.176-184**.**

7. Bak S. M. The existence and uniqueness of the global solution of the Cauchy problem for an infinite system of nonlinear oscillators on a two-dimensional lattice. *Math. and Comp. Modelling. Ser.: Phys. and Math. Sci.* 2011. Vol. 5. P. 3-9. (in Ukrainian)

8. Bak S. The existence of heteroclinic traveling waves in the discrete sine-Gordon equation with nonlinear interaction on a 2D-lattice. *Journal of mathematical physics, analysis, geometry*. 2018. Vol. 14, № 1. P. 16-26. doi:10.15407/mag14.01.016.

9. Bak S. M., Baranova O. O., Bilyk Yu. P. Correctness of the Cauchy problem for an infinite system of nonlinear oscillators on 2D-lattice. *Math. and Comp. Modelling. Ser.: Phys. and Math. Sci.* 2010. Vol. 4. P. 18-24.

10. Bak S. M., Kovtonyuk G. M. Existence of solitary traveling waves in Fermi-Pasta-Ulam system on 2D lattice. *Matematychni Studii*. 2018. Vol. 50, № 1. P.75-87. doi:10.15330/ms.50.1.75-87.

11. Bak S., Kovtonyuk G. Existence of standing waves in DNLS with saturable nonlinearity on 2D lattice. *Communications in Mathematical Analysis*. 2019. Vol. 22, № 2. P. 18–34.

12. Bak S. M., Kovtonyuk G. M. Existence of traveling waves in Fermi-Pasta-Ulam type systems on 2D-lattice. *J. Math. Sci.* 2021. Vol. 252, № 4 (January). P. 453-462. doi:10.1007/s10958-020-05173-6.

13. Bak S., N'Guerekata G., Pankov A. Well-posedness of initial value problem for discrete nonlinear wave equations. *Communications in Mathematical Analysis*. 2010. Vol. 8, № 1. P. 79–86.

14. Bak S. N., Pankov A. A. On the dynamical equations of a system of linearly coupled nonlinear oscillators. *Ukr. Math. J.* 2006. Vol. 58, № 6. P. 815-822. doi:10.1007/s11253-006-0105-6.

15. Bak S. N., Pankov A. A. Traveling waves in systems of oscillators on 2D-lattices. 2011. Vol. 174, № 4 (April). P. 916-920. doi:10.1007/s10958-011-0310-1.

16. Braun O. M., Kivshar Y. S. Nonlinear dynamics of the Frenkel-Kontorova model. *Physics Repts*. 1998. Vol. 306. P. 1-108.

17. Braun O. M., Kivshar Y. S. The Frenkel–Kontorova model. Berlin : Springer, 2004.

18. Daleckii Yu. L., Krein M. G. Stability of solutions of differential equations in Banach spaces. Providence: Amer. Math. Soc., 1974.

19. Fečkan M., Rothos V. Traveling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions. *Nonlinearity*. 2007. Vol. 20. P. 319-341.

20. Friesecke G., Matthies K. Geometric solitary waves in a 2D math–spring lattice. *Discrete and continuous dynamical systems*. 2003. Vol. 3, № 1 (February). P. 105–114.

21. MacKay R. S., Aubry S. Proof of existence of breathers for time–reversible a Hamiltonian networks of weakly coupled oscillators. *Nonlinearity*. 1994. Vol. 7. P. 1623–1643.

22. Pankov A. Traveling Waves and Periodic Oscillations in Fermi–Pasta–Ulam Lattices. London–Singapore : Imperial College Press, 2005. 196 p.

23. Pankov A. Gap solitons in periodic discrete nonlinear Shrödinger equation. *Nonlinearity*. 2006. Vol. 19. P. 27-40.

24. Pankov A. Gap solitons in periodic discrete nonlinear Shrödinger equation, II: Generalized Nehari manifold approach. *Discr. Cont. Dyn. Syst. A*. 2007. Vol. 19, № 2. P. 419–430.

25. Pankov A., Rothos V. Periodic and decaying solutions in DNLS with saturable nonlinearity. *Proc. Royal Society A.* 2008. Vol. 464. P. 3219–3236.

26. Srikanth P. On periodic motions of two-dimensional lattices. *Functional analysis with current applications in science, technology and industry*. 1998. Vol. 377. P. 118-122.