Sergiy Bak*, Dr. Sc.
Galyna Kovtonyuk**, Ph. D.
*Vinnytsia Mykhailo Kotsiubynskyi State Pedagogical University, Vinnytsia, Ukraine
e-mail: sergiy.bak@vspu.edu.ua
**Vinnytsia Mykhailo Kotsiubynskyi State Pedagogical University, Vinnytsia, Ukraine
e-mail: kovtonyukgm@vspu.edu.ua

ON WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR SYSTEM OF OSCILLATORS IN WEIGHTED SEQUENCE SPACES

Abstract. We consider an infinite system of ordinary differential equations that describes the dynamics of an infinite system of linearly coupled nonlinear oscillators on a two dimensional integer-valued lattice. We obtain the results on existence of a unique global solutions of the Cauchy problem in a wide class of weighted sequence spaces.

Key words and phrases: nonlinear oscillators, 2D-lattice, Cauchy problem, well-posedness, weighted sequence spaces.

We study equations that describe the dynamics of an infinite system of linearly coupled nonlinear oscillators on a two dimensional lattice. Let $q_{n,m} = q_{n,m}(t)$ be a generalized coordinate of the (n,m)-th oscillator at time t. It is assumed that each oscillator interacts linearly with its four nearest neighbors. The equations of motion of the system are of the form

$$\begin{gathered}\ddot{q}_{n,\,m} = a_{n-1,\,m}q_{n-1,\,m} + a_{n,\,m}q_{n+1,\,m} + b_{n,\,m-1}q_{n,\,m-1} + b_{n,\,m}q_{n,\,m+1} + \\ + c_{n,\,m}q_{n,\,m} - V'_{n,\,m}(q_{n,\,m}),\ (n,\,m) \in \mathbb{Z}^2.\end{gathered} \tag{1}$$

where $a_{n,\,m}, b_{n,\,m}, d_{n,\,m} \in \mathbb{R}$, $V_{n,\,m} \in C^1(\mathbb{R};\mathbb{R})$. We consider solutions of system (1) such that

$$\lim_{n,\,m\to\pm\infty} q_{n,\,m}(t) = 0, \tag{2}$$

i.e., the oscillators are at the rest at infinity.

We study the Cauchy problem for system (1) with initial conditions

$$q_{n,\,m}(0) = q^{(0)}_{n,\,m},\ \dot{q}_{n,\,m}(0) = q^{(1)}_{n,\,m},\ (n,\,m) \in \mathbb{Z}^2. \tag{3}$$

System (1) naturally can be considered as an operator-differential equation, namelly

$$\ddot{q} = Aq - B(q), \tag{4}$$

where $(Aq)_{n,\, m} = a_{n-1,\, m}q_{n-1,\, m} + a_{n,\, m}q_{n+1,\, m} + b_{n,\, m-1}q_{n,\, m-1} + b_{n,\, m}q_{n,\, m+1} + c_{n,\, m}q_{n,\, m}$, and the nonlinear operator B is defined by $(B(q))_{n,\, m} = V'_{n,\, m}\left(q_{n,\, m}\right)$, in the Hilbert, or even Banach, space of sequences.

We impose the following assumptions

(i) $\{a_{n,m}\}$, $\{b_{n,m}\}$ *and* $\{d_{n,m}\}$ *are bounded;*

(ii) $V_{n,\, m} \in C^1(\mathbb{R}; \mathbb{R})$, $V_{n,m}(0) = V'_{n,m}(0) = 0$, $(n,\, m) \in \mathbb{Z}^2$, *and* $V'_{n,\, m}$ *is locally Lipschitz continuous uniformly with respect to* $(n,\, m) \in \mathbb{Z}^2$, *i.e., for any* $R > 0$ *there exists a constant* $C = C(R) > 0$ *such that for all* $(n,m) \in \mathbb{Z}^2$:

$$\left|V'_{n,m}(r_1) - V'_{n,m}(r_2)\right| \le C|r_1 - r_2|, \quad |r_1|,|r_2| \le R.$$

Sometimes we use the following stronger than (ii) assumption

(ii') *assumption* (ii) *is satisfied with the constant* C *independent of* $R > 0$, *i.e.,* $V'_{n,\, m}$ *is globally Lipschitz continuous uniformly with respect to* $(n,\, m) \in \mathbb{Z}^2$.

Let $\Theta = \{\theta_{n,m}\}$ be a sequence of positive numbers (*weight*). We denote by l^2_Θ the space of all two-sided sequences $q = \{q_{n,m}\}$ of real numbers such that the norm

$$\|q\|_\Theta = \left(\sum_{(n,m)\in\mathbb{Z}^2} \theta_{n,m} q_{n,m}^2\right)^{\frac{1}{2}}.$$

is finite. This is a Hilbert space with the scalar product

$$(u,\, v)_\Theta = \sum_{(n,m)\in\mathbb{Z}^2} \theta_{n,m} u_{n,m} v_{n,m}.$$

We suppose that the weight $\Theta = \{\theta_{n,m}\}$ satisfies the following assumption

(iii) *the weight* Θ *be a regular, i.e., the sequence* $\{\theta_{n,m}\}$ *is bounded below by a positive constant and there exists a constant* $c_0 > 0$ *such that*

$$c_0^{-1} \le \frac{\theta_{n+1}}{\theta_n} \le c_0$$

for all $(n,m) \in \mathbb{Z}^2$.

Note that $l_\Theta^2 = l^2$ as $\theta_{n,m} \equiv 1$.

We obtain the following results.

Теорема 1. *Assume* (i), (ii') *and* (iii). *Then for every* $q^{(0)} \in l_\Theta^2$ *and* $q^{(1)} \in l_\Theta^2$ *problem (1), (3) has a unique global solution* $q \in C^2(\mathbb{R}; l_\Theta^2)$.

Теорема 2. *Assume* $(i)-(iii)$. *Suppose that the operator* A *is non-positive, i.e.,* $(Aq, q) \le 0$ *for all* $q \in l^2$. *Suppose also that one of the following two conditions holds:*

(a) $V_{n,m}(r) \ge 0$ *for all* $(n, m) \in \mathbb{Z}^2$ *and* $r \in \mathbb{R}$;

(b) *there exists a nondecreasing function* $h(\xi)$, $\xi \ge 0$, *such that* $\lim\limits_{\xi \to +\infty} h(\xi) = +\infty$ *and* $V_{n,m}(r) \ge h(|r|)$ *for all* $(n, m) \in \mathbb{Z}^2$ *and* $r \in \mathbb{R}$.

Then for every $q^{(0)} \in l_\Theta^2$ *and* $q^{(1)} \in l_\Theta^2$ *problem (1), (3) has a unique global solution* $q \in C^2(\mathbb{R}; l_\Theta^2)$.

Теорема 3. *Assume* (i) *and* (iii). *Suppose that* $V_{n,m}(r) = \dfrac{g_{n,m}}{p} r^p$, *where* $\{g_{n,m}\}$ *is a bounded sequence, and the operator* A *is negative definite in* l^2. *Then there exists* $\delta > 0$, *such that for any* $q^{(0)} \in l_\Theta^2$ *and* $q^{(1)} \in l_\Theta^2$ *with* $\|q^{(0)}\| \le \delta$ *and* $\|q^{(1)}\| \le \delta$ *problem (1), (3) has a unique global solution* $q \in C^2(\mathbb{R}; l_\Theta^2)$.

References

1. Bak S. M. Global well-posedness of the Cauchy problem for system of oscillators on 2D-lattice with power potentials. *Journal of Mathematical Sciences*. 2020. Vol. 246, No 5. P. 593-601.
2. Bak S. M. The existence and uniqueness of the global solution of the Cauchy problem for an infinite system of nonlinear oscillators on a two-dimensional lattice. *Math. and Comp. Modelling. Ser.: Phys. and Math. Sci.* 2011. Vol. 5. P. 3-9 (in Ukrainian).
3. Bak S. M., Baranova O. O., Bilyk Yu. P. Correctness of the Cauchy problem for an infinite system of nonlinear oscillators on 2D-lattice. *Math. and Comp. Modelling. Ser.: Phys. and Math. Sci.* 2010. Vol. 4. P. 18–24 (in Ukrainian).
4. Bak S., N'Guerekata G. M., Pankov A. Well-posedness of initial value problem for discrete nonlinear wave equations. *Commun. Math. Analysis*. 2010. Vol. 8. No 1. P. 79–86.
5. Bak S. N., Pankov A. A. On the dynamical equations of a system of linearly coupled nonlinear oscillators. *Ukr. Math. J.* 2006. Vol. 58, No 6. P. 815–822.